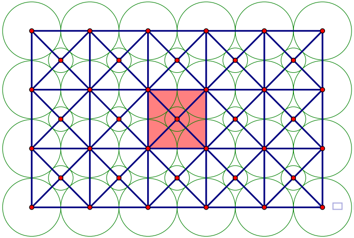
What are Connected?

1. László Fejes Tóth's work on maximum density packing density circular disks in the plane.
2. Flip and Flow. See the talk by S. Gortler.
3. Thomas Fernique's circle packing improves László Fejes Tóth's best guesses for given radius ratios.
4. Sticky packing disks are discussed in the talk by Louis Theran.
5. Veit Elser's problem is to maximize the the sum of the radii for a fixed number of circles in a container.
6. Sándor Fekete and others take a finite number of circles as given, and try to pack them into a given container.

Part 1: History

"Axel Thue provided the first proof that this was optimal in 1890, showing that the hexagonal lattice is the densest of all possible circle packings, both regular and irregular. However, his proof was considered by some to be incomplete. The first rigorous proof is attributed to László Fejes Tóth in 1940." (From Wikipedia).

But Fejes Tóth did more: He found a whole series of packings with a range of radii that he proposed as candidates for the most dense packings given a particular ratio of the radii. For example, if there are just two radii, with a ratio of $\sqrt{2} - 1 = 0.414\ldots$ Aladar Heppes (2000) proved that the following configuration is the most dense at $\delta = \pi(2 - \sqrt{2})/2 = 0.92015\ldots.$



Compact Packings = Triangulated Packings

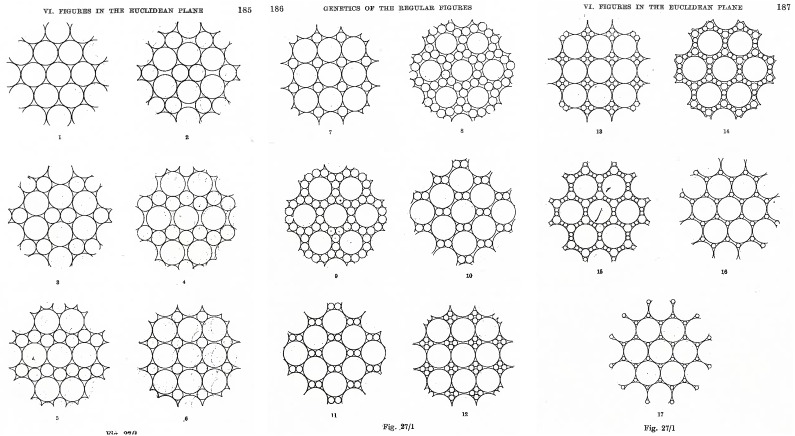
The *graph of a circle packing* is obtained by connecting the centers of each pair of touching disks by a line segment. When that graph of a packing is a triangulation, Fejes Tóth called the packing a *compact packing*. In many cases such "compact packings" were candidates for the most dense packings with those radii sizes. For two sizes of radii the following are his candidates for the maximum density.

Most packings that come up with respect to density questions are periodic, and so it is natural to simply assume that they are periodic, which is the same as assuming they live on a (flat) torus.

Any triangulated packing with different sized radii has a density strictly greater than $\pi/\sqrt{12} = 0.90689968..$, the density of the standard triangulated packing with all disks the same radii.

Fejes Toth's packings

From “Regular Figures” by László Fejes Tóth



Fejes Toth's density estimates

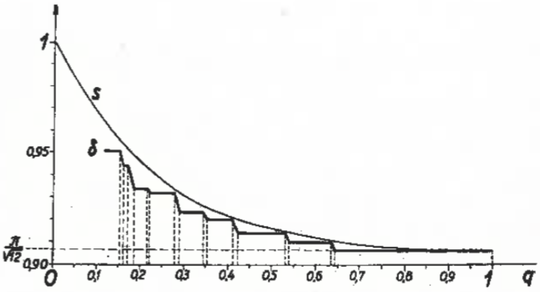


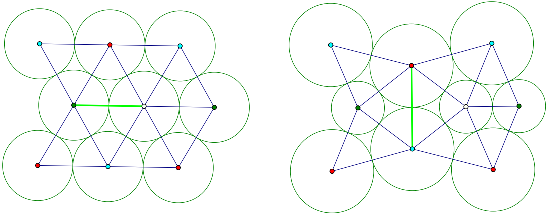
Fig. 27/3

The function s above is an upper bound on the density, by August Florian (1960), where ρ is the minimum ratio r_i/r_j of the circle radii.

$$s = \frac{\pi\rho^2 + 2(1-\rho^2)\arcsin(\frac{\rho}{1+\rho})}{2\rho\sqrt{2\rho+1}}.$$

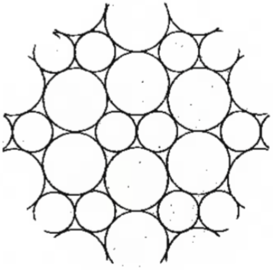
Part 2: Edge flipping

Another method is to *flip* an edge of the triangulation, which is where an edge is removed and replaced as the other diagonal in the resulting quadrilateral. Any edge can be flipped, as long as you don't flip away a degree 3 vertex.



The packing on the right is obtained by flipping the green edge on the 4 vertex lattice triangulation. The symmetry of the abstract triangulation and the uniqueness of the packing implies that the symmetry persists in the metric configuration of the packing, which implies that there are just two radii in this flipped packing. This is László Fejes Tóth's packing #3. It has radius ratio 0.6375559772 . . ..

Part 2: Continuous Edge flipping: László Fejes Tóth



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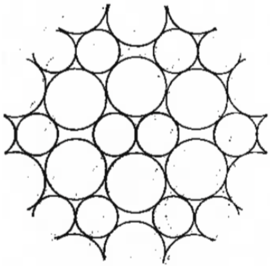

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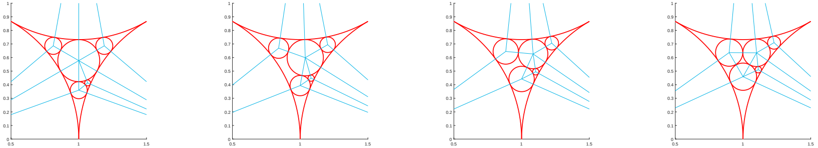
László Fejes Tóth's idea was to do an edge flip motion continuously, which increased the radius ratio, but decreased the density, until the density was $\pi/\sqrt{12}$. The above packing #2 was his best guess of the limit of circle packings with density greater than $\pi/\sqrt{12}$, and radius ratio as large as possible. The limiting radius ratio here is $0.6457072159\ldots$.

Alladár Heppes showed that if there are just two radii, $0.63755\ldots$ and 1, then the most dense packing in the plane is the one above on the left.

Part 2: Flip and Flow: Thurston

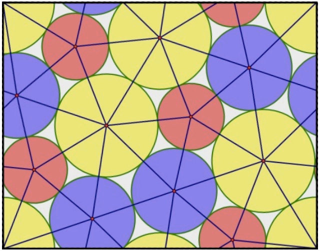
Edge flipping can be proved to be done continuously always, at least in the case of triangulations of a sphere. Start with any triangulation of the sphere with n disks that can be achieved with a packing and then continuously move through packings, the flow, to achieve the flipped packing on n disks. Any triangulation can be created this way. (See S. Gortler's talk)

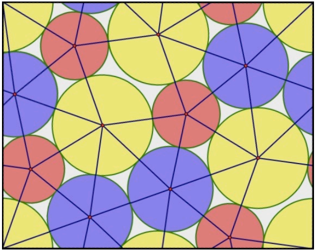


(a) (b) (c) (d)

Part 3-Fernique's Packing

Previously Tom Kennedy had found all 9 of the triangulated packings that had 2 different radii. Recently Thomas Fernique classified all the periodic triangulated packings that had 3 different radii. One particular packing below stood out.

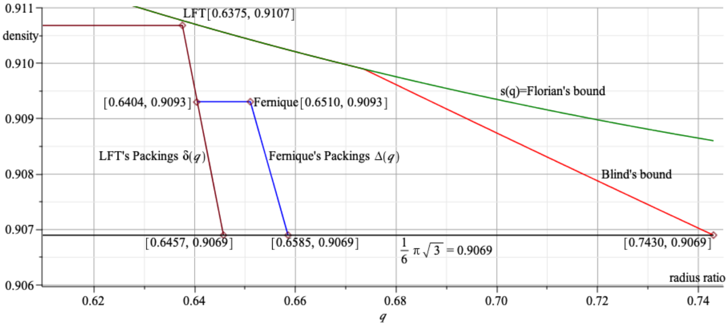




Fernique's triangulated packing on the left has the ratio of the smallest over the largest radius as 0.6510501858 . . . . When a little flip-flow is done to bring the density down to $\pi/\sqrt{12}$, it has radius ratio 0.6585340820 . . . as in the packing on the right.

Part 3-Plotting Fernique

This is the addendum to László Fejes Tóth's density plot above. This is on the extreme right.



Part 3-Conjecture

It known by G. Blind that any packing in the plane with radius ratio between 0.74299.. and 1, has density no larger than $\pi/\sqrt{12}$.

Conjecture

Any triangulated circle packing in the plane with different radii has radius ratio at most $0.6510501858\ldots$.

More ambitiously

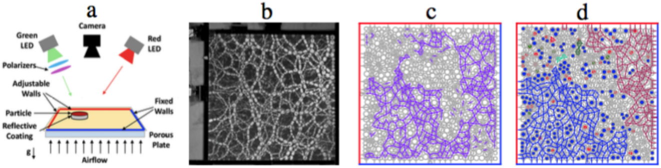
Conjecture

Any circle packing in the plane with radius ratio between $0.6585340820\ldots$ *and* 1 *has density at most* $\pi/\sqrt{12}$.

See the talk by Zhen Zhang.

Part 4-Isostatic Conjecture-Granular Materials

One of my motivations for studying the rigidity of circle packings is from the study of granular materials. Laman's theorem, etc. comes up, although it does not apply without more work. The following is a picture of the experimental set-up to study some of the rigidity behavior in Liu et. al. on the ArXiv.



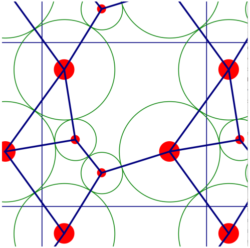
The connection: They refer to the Lee and Streinu (k, l) pebble game to help model friction.

Part 4-Isostatic Conjecture

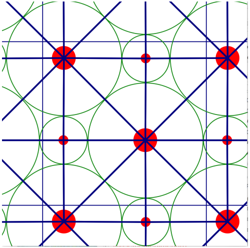
Conjecture (Isostatic Conjecture)

If circular disks packed rigidly in a "container" with generic radii, then there is only a one-dimensional equilibrium stress, and the corresponding minimal number of contacts.

Examples:



An isostatic packing with 4 disks and 7 edges.



A non-isostatic packing with 4 disks and 12 edges.

Part 4-The Isostatic Theorem

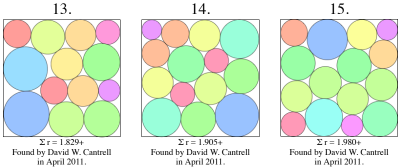
Theorem (Connelly, Solomonides, and Yampolskaya, 2017)

If a jammed packing $\mathcal{P}_0$ with n vertices in a torus $\mathbb{T}^2$ is chosen so that the ratio of packing disks, and torus lattice Λ_0, is generic, then the number of contacts in $\mathcal{P}_0$ is $2n - 1$, and the packing graph is isostatic.

The idea is that the dimension of the space of packings, with the inversive distances replacing edge lengths, is only consistent with the minimal number of contacts given by isostatic condition.
See the talk by Louis Theran about isostatic packings in a tricusp, and a more direct proof, using “sticky disks”.

Part 5-Maximizing the sum of radii

Veit Elser (talking on Tuesday) has been interested in the problem of maximizing the sum of n radii of circles in a unit square.



Observation/Comment: As n goes to infinity the sum of the n radii, divided by $\sqrt{n}$ should? approach $1/\sqrt{\sqrt{12}} = 0.53728\ldots$ This is the limit when all the radii are equal.

Part 6-The packing puzzle

Suppose a container, square, circle, triangle, or torus, is given and you have a finite collection of circular disks (or other shapes), when can you pack that container? For example, here is Aaron Becker's youngest, Lewis having a hard time for circles in a circle. Thanks to Sándor Fekete for the video featuring this

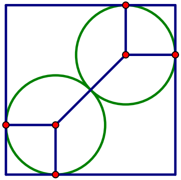
https://www.ibr.cs.tu-bs.de/users/fekete/Videos/PackingCirclesInSquares.mp4

Part 6-The packing puzzle

Theorem (Demaine, Fekete, Lang)

It is always possible to pack a set of circular disks into a square if and only if the expected density is at most $\pi(3-\sqrt{2}) = 0.53901208...$

The following is the critical case. For a unit square, the radius is $(2-\sqrt{2})/2 = 0.2928932188...$



Notice that there is no consideration about the distribution of radii. Given that a set of disks do pack some shape, finding the packing can be hard.

Part 6-A Packing Game

Question

Suppose a collection of circular disks and a container is given, what is the most dense arrangement they can form in some scaled form of the the container?

Example: Suppose the packing disks have radii 1 and $0.63755\ldots$ in proportion 1 and x. One guess is to have two sets, one with equal numbers of the two sizes whose density is 0.910683.. and the rest with the hexagonal packing of all the same size with density 0.906899... The resulting density will be an average as a function of x of those densities that has a maximum when they are in equal proportions, by Heppes's result.